An Intersecting Loop Model as a Solvable Super Spin Chain

M. J. Martins[1,2] , B. Nienhuis[1] and R. Rietman[1] *

[1] *Instituut voor Theoretische Fysica, Universiteit van Amsterdam*

Valcknierstraat 65, 1018 XE Amsterdam, The Netherlands

and

[2] *Departamento de Física, Universidade Federal de São Carlos*

Caixa Postal 676, 13565-905, São Carlos, Brasil

Abstract

In this paper we investigate an integrable loop model and its connection with a supersymmetric spin chain. The Bethe Ansatz solution allows us to study some properties of the ground state. When the loop fugacity q lies in the physical regime, we conjecture that the central charge is $c = q - 1$ for q integer < 2. Low-lying excitations are examined, supporting a superdiffusive behavior for $q = 1$. We argue that these systems are interesting examples of integrable lattice models realizing $c \leq 0$ conformal field theories.

PACS numbers: 05.20-y, 0.5.50+q, 04.20.Jb

*Present address: Philips Research Laboratories, Prof. Holstlann 4, 5656 Eindhoven, The Netherlands

In Statistical Mechanics the basic difference between an ordinary local model (vertex models, spin chain systems) and a loop model is that for the latter the total weight configuration cannot be written as a product of local weights. Being intrinsically non-local, loop models appears as ideal paradigm for studying statistical properties of extended objects such as polymers (see e.g. refs. [1]).

In this letter we investigate some critical properties of an integrable intersecting loop model in a two dimensional square lattice [2]. The fact that intersections between the polygons configurations are allowed makes this loop model very interesting. In this case it is not clear at all how to find a transformation to a standard local model [3, 4]. The model can be considered as a Lorentz gas of particles moving through a set of scatters randomly distributed in the nodes of the two dimensional square lattice [5, 6, 7]. The scatters are double-sided mirrors allowing right-angle reflections and they are placed along the diagonals of the square lattice. The particles move along the edges of the lattice and arriving on a node can be scattered to the left, to the right or pass freely in the case of absence of a scatter. We denote by w_a, w_b and w_c the Boltzmann weights corresponding to these three possibilities, respectively. By imposing periodic boundary conditions, each particle follows a closed path. If for every closed loop we assign a fugacity q, then the partition function Z can be written as

$$Z = \sum_{scatter\ configurations} w_a^{n_a} w_b^{n_b} w_c^{n_c} q^{\#paths} \tag{1}$$

where n_a,n_b and n_c are the number of weights w_a, w_b and w_c appearing in a configuration, respectively. We noticed that strictly when $w_c = 0$ the closed loops configurations no longer intersect. In this limit the partition function (1) can be seen as a graphical representation of the critical q^2-state Potts model [3].

Despite the inherent non-locality of this loop model it is still possible to formulate a purely local condition that two different transfer-matrices commute for arbitrary system size. This is a sufficient condition for integrability and it imposes a restriction on the manifold of possible weights w_a, w_b and w_c configurations. It turns out that the intersecting loop model is integrable

[2] if the Boltzmann weights are parametrized as follows

$$w_a(\mu) = 1 - \mu, \quad w_b(\mu) = \mu, \quad w_c(\mu) = (1 - \frac{q}{2})\mu(1 - \mu) \tag{2}$$

Clearly, the interesting physical regime is when $q < 2$, where all the "probabilities" w_a, w_b and w_b can be made positive for $0 < \mu < 1$. Anyway, it is interesting to noticed that when q is an integer ≥ 2 these weights reproduce (after a rescaling of μ) the ones appearing in the S-matrix solution of the $O(q)$ invariant non-linear sigma-model [8]. This strongly suggest that a mapping onto a ordinary local vertex model should not be completely ruled out, at least for some values of q. Indeed, here we argue that when $q \in \mathbf{Z}$ the intersecting loop model can be realized in terms of a local spin chain which is invariant by the superorthosympletic $Osp(n|2m)$ superalgebra. This is an important observation, since it will allow us to study the physical regime of (2) for q integer < 2. We then use the fact that the $Osp(n|2m)$ super spin chain is solvable by the Bethe Ansatz in order to find that their critical properties are governed by $c \leq 0$ conformal field theories. We remark that $c \leq 0$ theories appear to be useful in condensed matter systems such as the quantum Hall effect [9, 10], disordered models [11] and polymer field theories [12]. This means that our models could be relevant lattice paradigms for some condensed matter applications, and being integrable, they might provide further physical insight as well.

Essential to our approach is to observe that the weights (2) can be derived in the context of a standard Yang-Baxter solution for a local vertex model. These weights are in one-to-one correspondence to the generators of a degenerated point of the Birman–Wenzel–Murakami algebra [13]. This algebra is generated by the identity I_i, a braid b_i and a Temperely–Lieb operator E_i acting on sites i and $i+1$ of a quantum spin chain of length L. On the degenerate point the braid operator becomes a generator of the symmetric group, namely

$$b_i b_{i\pm1} b_i = b_{i\pm1} b_i b_{i\pm1}, \quad b_i^2 = I_i, \quad b_i b_j = b_j b_i \text{ if } |i - j| \geq 2 \tag{3}$$

and the other set of relations closing the degenerated point of the braid-monoid algebra (see e.g ref. [15]) are

$$E_i E_{i\pm1} E_i = E_i, \quad E_i^2 = q E_i, \quad E_i E_j = E_j E_i \text{ if } |i - j| \geq 2 \tag{4}$$

and

$$b_iE_i = E_ib_i = E_i, \quad E_ib_{i\pm1}b_i = b_{i\pm1}b_iE_{i\pm1} = E_iE_{i\pm1} \tag{5}$$

It is not difficult to see that relations (3-5) can be baxterized [14, 15], providing us with a rational solution of the Yang-Baxter equation having precisely the weights w_a, w_b and w_c. To make further progress we search for a representation of the algebraic relations (3-5). At least for integer q, such representation can be found in terms of the invariants of the superalgebra $Osp(n|2m)$ [14, 15]. This superalgebra combines the $O(n)$ symmetry and the simpletic $Sp(2m)$ algebra, and the integers n and $2m$ play the role of the number of bosonic and fermionic degrees of freedom (see e.g. ref. [16]). The braid operator b_i becomes the graded permutation between the $(n+2m)$ degrees of freedom which is defined by [17]

$$b_i = \sum_{a,b=1}^{n+2m} (-1)^{p(a)p(b)} e_{ac} \otimes e_{bd} \tag{6}$$

where $p(a)$ is the parity distinguishing the bosonic ($p(a)=0$ for $a=1,\cdots,n$) and the fermionic ($p(a)=1$ for $a=n+1,\cdots,n+2m$) elements. Explicit expressions for the monoids E_i have been recently discussed in detail in ref. [15]. The important point which matters here, however, is that the fugacity q is precisely the difference between the number of bosonic and fermionic degrees of freedom. More precisely, we have

$$q = n - 2m \tag{7}$$

The formulation of the corresponding transfer-matrix has to respect the bosonic and the fermionic graduations [17, 18]. This is possible by writing $T(\lambda)$ as the supertrace of an auxiliary monodromy operator, $T(\lambda) = \sum_{a\in\mathcal{A}} (-1)^{p(a)} \mathcal{T}_{aa}$, where $\mathcal{A}$ stands for the horizontal space of $(n+2m)$ variables of the vertex model. As usual the monodromy matrix is composed by a product of vertex operators $\mathcal{L}_{\mathcal{A}i}(\lambda)$ which are given by

$$\mathcal{L}_{\mathcal{A}i}(\lambda) = (1 - \frac{q}{2} - \lambda)b_i + \lambda(1 - \frac{q}{2} - \lambda)I_i + \lambda E_i \tag{8}$$

Performing the scale $\lambda \to \mu(1-q/2)$ in equation (8), it is straightforward to see the correspondence between the weights (2) and the operators I_i, b_i and E_i. The corresponding

local $Osp(n|2m)$ spin chain $\mathcal{H}$ is proportional to the logarithmic derivative of the transfer matrix around the regular point $\lambda = 0$ where the vertex operator reduces to the graded permutation [17, 18]. The Hamiltonian can then be expressed in terms of the operators b_i and E_i as

$$\mathcal{H} = \pm \sum_{i=1}^{L} \left\{ b_i + \frac{1}{1-q/2} E_i \right\} \tag{9}$$

where the sign in (9) is chosen to select the antiferromagnetic regime of the theory. This supersymmetric Hamiltonian, with periodic boundary conditions imposed, admits a Bethe Ansatz solution . This means that the eigenenergies $E(L)$ on a ring of size L are parametrized in terms of complex set of variables $\{\lambda_j^a\}$, satisfying coupled non-linear Bethe Ansatz equations. These equations are equivalent to the analyticity of the eigenvalues of the corresponding transfer matrix and also reflect the underlying $Osp(n|2m)$ group symmetry. By taking into account these properties (analytical Bethe Ansatz approach) we can derive that they are given by

$$\left[\frac{\lambda_j^{(a)} - i\frac{\delta_{a,1}}{2\eta_a}}{\lambda_j^{(a)} + i\frac{\delta_{a,1}}{2\eta_a}} \right]^L = \prod_{b=1}^{r} \prod_{k=1,\, k\neq j}^{m_b} \frac{\lambda_j^{(a)} - \lambda_k^{(b)} - i\frac{C_{a,b}}{2\eta_a}}{\lambda_j^{(a)} - \lambda_k^{(b)} + i\frac{C_{a,b}}{2\eta_a}}, \quad j = 1, \ldots, m_a; \quad a = 1, \ldots, r \tag{10}$$

and the eigenenergies are parametrized by $\{\lambda_j^{(1)}\}$

$$E(L) = -\sum_{i=1}^{m_1} \frac{1}{[\lambda_i^{(1)}]^2 + 1/4} + L \tag{11}$$

where C_{ab} is the Cartan matrix, r is the number of roots and η_a is the normalized length of the a-th root of the $Osp(n|2m)$ superalgebra. We recall that the Dynkin diagrams of the models with $n = 1$ are special. Consequently a peculiar two-body phase shift occurs for the last root $\{\lambda_j^{2m}\}$. For an algebraic Bethe Ansatz derivation of equations (10,11) for some classes of $Osp(n|2m)$ models as well as for further technical details we refer to ref. [15].

We now turn to the study of the critical behaviour of the super spin chain (9). The existence of a Bethe Ansatz solution allows us, in principle, to calculate the eigenvalues $E(L)$ for quite large values of L, providing us with reasonable estimates of the finite size effects. For a conformally invariant system, the universality can then be determined by exploiting a set of important relations satisfied by the eigenvalues on a strip of size L [19]. For example, the

central charge c is related to the ground state energy $E_0(L)$ by [20]

$$\frac{E_0(L)}{L} = e_\infty - \frac{\pi v_s c}{6L^2} + O(L^{-2}) \tag{12}$$

where e_∞ is the ground state energy per particle in the thermodynamic limit and v_s is the sound velocity. These two parameters can be determined exactly from the unitarity and the crossing properties (around $\lambda = 1 - q/2$) of the transfer matrix $T(\lambda)$. In fact, these properties together imply that, in the thermodynamic limit, the largest eigenvalue $\Lambda_0(\lambda)$ of the transfer matrix satisfies the relations

$$\Lambda_0(\lambda)\Lambda_0(-\lambda) = [1-\lambda^2][(1-q/2)^2 - \lambda^2], \quad \Lambda_0(\lambda) = \Lambda_0(1-q/2-\lambda) \tag{13}$$

Solving this equations with the restriction that the solution is free of zeroes in the physical strip $0 < \lambda < 1 - q/2$ and taking its logarithmic derivative at $\lambda = 0$, we find that

$$e_\infty = -\frac{1}{1-q/2}\left\{\psi(\frac{1}{2} + \frac{1}{2-q}) - \psi(\frac{1}{2-q}) + 2\ln(2)\right\} + 1 \tag{14}$$

where $\psi(x)$ is the Euler function. The sound velocity measures how the energy scales with the low momenta. If we recall that equations (13) are identical to the one we solve to find the crossing factors in a relativistic S-matrix theory, we can obtain that the appropriate relativistic scale is given by

$$v_s = \frac{\pi}{1-q/2} \tag{15}$$

We now have the basic ingredients to begin our analysis of the finite size effects for the ground state energy. Let us first consider the case when the fugacity is one . For this value of q, the partition function of the intersecting loop model is trivial (constant) and therefore $E_0(L)/L$ is exactly e_∞ for any size L. In other words, all the finite size corrections to the ground state are null, and in particular $c = 0$. However, from the spin chain point of view, this scenario is far from being trivial, and provides us with an important check concerning the loop model $\leftrightarrow$ spin chain mapping. The simplest spin chain giving us $q = 1$ is the $Osp(3|2)$ model. Its spectrum is given in terms of one level nested Bethe Ansatz and the Bethe equations

are parametrized by two sets of variables $\{\lambda_j^{(1)}, \lambda_j^{(2)}\}$. The ground state is characterized by a complex root distribution, forming "fractional" strings of the following type

$$\lambda_j^{(1)} = \xi_j^{(1)} \pm i/4 + O(e^{-L}), \quad \lambda_j^{(2)} = \xi_j^{(2)} \pm i/4 + O(e^{-L}) \tag{16}$$

By solving numerically the corresponding Bethe Ansatz equation for some values of L and substituting the value of $\{\lambda_j^{(1)}\}$ in equation (11), it is remarkable to see how the Bethe Ansatz roots conspire together in order to produce the simple result $E_0(L) = -3L$ exactly. Note that for this model that $e_\infty = -3$ (see equation (14)). Although similar effect has been observed before in fine tuned anisotropic spin chains [21], to the best of our knowledge, this is the first time that such simplification is noted in a free-parameter (isotropic) set of Bethe Ansatz equations. This gives confidence to investigate the super spin chains for other values of $q < 2$.

For $q \neq 1$, the procedure described above can also be used, and we have analysed the equations (10,11) for sizes up to $L = 80$. In table 1 we show our estimates for the central charge c for the $Osp(2|2)$, $Osp(1|2)$, $Osp(1|4)$ and $Osp(1|6)$ supersymmetric spin chains, corresponding to the values $q = 0, -1, -3$, and -5, respectively. In our numerical analysis we already have considered the presence of logarithmic contributions of $O(1/L^2 \ln(L))$ to the finite size corrections of the ground state. We remark that this kind of correction typically cannot be overcome by standard transfer matrix or Hamiltonian diagonalization due to size limitations. All the results lead us to the following conjecture for the central charge behaviour for these models when q is a integer < 2

$$c = q - 1 \tag{17}$$

This formula also reproduces the central charge in the limit $q = 2$. For this point, the weight w_c vanishes and hence we expect the critical behaviour of the isotropic 6-vertex model. Furthermore, the ground state of the $Osp(1|2n)$ models ($q = 1 - 2n$) are parametrized by real roots, and by using an analytical method developed in ref. [23] we obtain $c = -2n$, in accordance with equation (17). Finally, it is interesting to note that formula (17) can be derived in the context of a super Sugawara construction of the Virasoro algebra developed by Goddard et al [24]. Following this work the central charge at certain level k is $c = ksdimG/(k + Q_G/2)$,

where sdimG and Q_G is the superdimension and the value of the quadratic Casimir of the superalgebra G. This latter expression applied to the $Osp(n|2m)$ superalgebra with $k = 1$ gives $c = n - 2m - 1$, which agrees with the formula we proposed for the central charge of the loop model when $q = n - 2m \leq 2$.

In principle we can use a similar approach to study the excitations via numerical or analytical analysis of the Bethe Ansatz equations (10,11). This study is of particular physical relevance for the Lorentz lattice gas $q = 1$ model . In this case an interesting quantity is the fractal dimension d_f of the loops, which characterizes the diffusion properties of the particles through the lattice [5, 7]. Recent numerical simulations [7] predicted a superdiffusive behaviour $d_f = 2$ as long as the density of mirrors is smaller than one ($w_c \neq 0$). To lend some theoretical support to this class of universality we have studied the finite size corrections for the lowest excitation present in the $q = 1$ model. We find that this excitation is of spin wave type, made by taking out one root λ_j^2 from the ground state configuration. In table 2 we present the finite size sequences for the exponent $x = 2h$, where h is the conformal weight. We see that the extrapolated value for h is very small, indicating the presence of a zero conformal weight and consequently predicting $d_f = 2 - 2h = 2$ whithin reasonable precision. We remark that better numerical data was prevented by strong logarithmic corrections [7].

A second example where the study of excitations plays an important role is for the supersymmetric $Osp(1|2)$ model ($q = -1$). Its spectrum is parametrized by a single non-linear equation and some low-lying states are amenable of analytical study (real Bethe Ansatz roots). This model has $c = -2$ and to obtain the Ramond sector, having lowest conformal dimension $h = -1/8$, we need to impose antiperiodic boundary condition on the even $U(1)$ sectors of the superspin chain [22]. The twisted Hamiltonian still has a continuous $SU(2)$ symmetry and the $U(1)$ charge plays the role of a fermionic index . In principle, there is no inconsistency to interpret the twisted sector as an exited state, even though its energy lies below the ground state, because the super spin chain is indeed non hermitian (though the eigenvalues are real). This scenario is in remarkably similar to the one proposed in refs. [9, 10] to explain certain properties of edge excitations in the fractional Hall effect. Furthermore, concerning the eigen-

spectrum of the primary fields, the twisted spin chain could either be seen as a $c = -2$ theory or as $c = 1$ Coulomb gas with radius of compactification $\rho = \sqrt{2}$. We also noticed that the $Osp(2|2)$ spin chain is the prototype for realizing both $c = -1$ and $c = 2$ field theories. Again, our numerical data supportes the presence of a field with dimension $h = -1/8$ in the twisted spin chain. Very recently, such relations between $c \leq 0$ and $c > 0$ have been proposed and discussed in the literature [25]. This suggests that the lattice models discussed in this paper are realizations of non-unitary ($c < 0$) conformal theories that are related (via appropriate twisting) to $c > 0$ systems. In particular our numerical result for the twisted $q = 1$ model indicates central charge $c = 3$, suggesting that the continuum limit of the Lorentz gas might be governed by a $N = 2$ supersymmetric field theory (see e.g. ref.[26]) . It remains to be seen, however, what kind of physical information can be obtained from this observation.

In summary, a solvable loop model has been mapped onto a super spin chain and we have found its central charge behaviour for integer values of the fugacity in the physical regime. We hope that our discussions lend support to the idea that these models appears to be the ideal lattice candidates for describing conformal properties of relevant condensed matter systems.

Acknowledgements

It is a pleasure to thank J.de Gier for may useful discussions. This work was support by FOM (Fundamental Onderzoek der Materie) and Fapesp (Fundação de Amparo á Pesquisa do Estado de S. Paulo), and was partially done in the frame of Associate Membership programme of the International Centre for Theoretical Physics, Trieste Italy.

References

[1] B. Nienhuis, *Int.J.Mod.Phys. B 4 (1990) 929*

[2] B. Nienhuis and R. Rietman, *A solvable loop model with intersections, preprint IFTA-92-35* ; R. Rietman, *Yang–Baxter equations, Hyperlatices and a loop model, PhD-thesis (unpublished)*

[3] R.J. Baxter, S.B. Kelland and F.Y. Wu, *J.Phys.A:Math.Gen. 9 (1976) 397*

[4] B. Nienhuis, S.O. Warnaar and H.J. Blöte, *J.Phys.A:Math.Gen. 26 (1993) 477*

[5] T.W. Ruijgrok and E.G.D. Cohen, *Phys.Lett.A 133 (1989) 415*

[6] R.M. Ziff, X.P. Kong and E.G.D. Cohen, *Phys.Rev.A 44 (1991) 2410*

[7] E.G.D. Cohen and F. Wang, *J.Stat.Phys. 81 (1995) 445* ; Meng-She Cao and E.G.D. Cohen, *J.Stat.Phys. 87 (1997) 147*

[8] A.B. Zamolodchikov and Al.B. Zamolodchikov , *Ann.Phys. 120 (1979) 253*

[9] X.G. Wen and Y.S. Wu *Nucl.Phys.B 419 (1994) 455* ; X.G. Wen, Y.S. Wu and T. Hatsugai, *ibid. 422 (1994) 476*

[10] M. Milovanovic and N. Read, *Phys.Rev.B 53 (1996) 13559*

[11] A.W.W. Ludwig, M.P.A. Fischer, R. Shankar and G. Grinstein, *Phys.Rev.B 50 (1994) 7526* ; C. Mudry, C. Chamon and X.G. Wen, *Nucl.Phys.B 466 (1996) 383* ; D. Bernard, *“ (Perturbed) Conformal Field Theory applied to 2D Disordered Systems: An Introduction”, Cargese Lectures, hepth/9509137*; Z. Maassarani and D. Serban, *Nucl.Phys.B 489 (1997) 603*

[12] H. Saleur, *Nucl.Phys.B 382 (1992) 486*

[13] M. Wadati, T. Deguchi and Y. Akutsu, *Phys.Rep. 180 (1987) 247, and references there in*

[14] M.J. Martins and P.B. Ramos, *J.Phys.A:Math.Gen. 27 (1994) L703*

[15] M.J. Martins and P.B. Ramos, *Nucl.Phys.B 500 (1997) 579*

[16] J.F. Cornwell, *Group Theory in Physics, Vol. 3 , Academic Press, 1989*

[17] P.P. Kulish and E.K. Sklyanin , *J.Sov.Math. 19 (1982) 1596* ; P.P. Kulish, *ibid. 35 (1986) 2648*

[18] F.H.L. Essler and V. Korepin, *Phys.Rev.B 46 (1992) 9147*

[19] J.L. Cardy, *Phase Transitions and Critical Phenomena* edited by C. Domb and J.L. Lebowitz (Academic, New York, 1987), Vol.11, p.55

[20] H.Blöte, J.L. Cardy and M.P. Nightingale, *Phys.Rev.Lett. 56 (1996) 742*; I. Affleck, *ibid. 56 (1996) 746*

[21] F.C. Alcaraz, M.N. Barber and M.T. Batchelor, *Ann.Phys. 182 (1988) 280*

[22] M.J. Martins, *Nucl.Phys.B 450 (1995) 768*

[23] H.J. de Vega and F. Woynarowich, *Nucl.Phys.B 251 (1985) 439* ; F. Woynarovich and H.P. Eckle, *J.Phys.A:Math.Gen. 20 (1987) L97*

[24] P. Goddard, D. Olive and G. Waterson, *Commun.Math.Phys. 112 (1987) 591*

[25] H. Kausch, *“ Curiosities at c=-2” ,hepth/9510149* ; S.Guruswamy and A.W.W. Ludwig, *“ Relating* $c < 0$ *and* $c > 0$ *Conformal Field Theories”, hep-th/9612172*; V. Gurarie, M. Flohr and C. Nayak, *Nucl.Phys.B 498 (1997) 513* ; J.C. Lee and X.G. Wen, *Electron and Quasiparticle Exponents of Haldane-Rezayi state in Non-abelian Fractional Quantum Hall Theory, cond-mat/9705303*

[26] G. Mussardo, G. Sotkov and M. Stanishkov, *Int.J.Mod.Phys. A4 (1989) 1135*

L	$Osp(2\|2)$	$Osp(1\|2)$	$Osp(1\|4)$	$Osp(1\|6)$
16	-1.11540	-1.92337	-4.08344	-6.52566
24	-1.10037	-1.93829	-4.02750	-6.25291
32	-1.09075	-1.94574	-4.00716	-6.15381
40	-1.08401	-1.95041	-3.99753	-6.10483
48	-1.07896	-1.95369	-3.99228	-6.07658
56	-1.07500	-1.95617	-3.98917	-6.05863
64	-1.07179	-1.95812	-3.98721	-6.04643
72	-1.06911	-1.95973	-3.98593	-6.03771
80	-1.06683	-1.96107	-3.98508	-6.03125
Extr.	-1.01 (1)	-1.996 (2)	-3.985 (3)	-5.997 (1)

Table 1: Finite size sequences for the extrapolation of the central charge for the models $Osp(2|2)$ ($q=0$), $Osp(1|2)$ ($q=-1$), $Osp(1|4)$ ($q=-3$) and $Osp(1|6)$ ($q=-5$).

Table 2: Finite size sequences for the extrapolation of the lowest conformal dimension $x=2h$ of the $q=1$ model.

L	$x = 2h$
16	$8.464651\ 10^{-2}$
32	$6.764093\ 10^{-2}$
48	$6.032298\ 10^{-2}$
64	$5.592822\ 10^{-2}$
80	$5.288533\ 10^{-2}$
96	$5.060282\ 10^{-2}$
112	$4.88002\ 10^{-2}$
128	$4.73245\ 10^{-2}$
Extr.	$1.33\ 10^{-2}$